

A. Standard exercises:

8.1 We consider the following functions: $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = x^2y + 2x^2 - 2xy - 4x + y \quad \text{and} \quad g(x, y, z) = 2xy - 3yz.$$

- (a) For which values of $c \in \mathbb{R}$ is the level curve $f^{-1}(c)$ a submanifold of \mathbb{R}^2 (and of what dimension)?
- (b) For which values of $c \in \mathbb{R}$ is the level surface $g^{-1}(c)$ a submanifold of \mathbb{R}^3 (and of what dimension)?

8.2 (a) Let $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ be a regular point of the surface S defined by the equation $f(x, y, z) = 0$. Prove that the tangent vector plane T_pS is the plane orthogonal to the gradient $\vec{\nabla}f(p)$.

- (b) The affine tangent plane to a surface S at a regular point p is the set of points of \mathbb{R}^3 such that the vector $\vec{pq} \in T_pS$. Show that the affine tangent plane is given by

$$A_pS = \{q \in \mathbb{R}^3 \mid \langle q - p, \vec{\nabla}f(p) \rangle = 0\}.$$

- (c) By applying the previous result, obtain the formula giving the first-order approximation of a differentiable function of two variables $z = \varphi(x, y)$ in the neighborhood of a point (x_0, y_0) (Taylor expansion of order 1).

8.3 Show that the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface (i.e. a submanifold of dimension 2) and compute its affine tangent plane at a point $p = (x_0, y_0, z_0)$.

8.4 Two differentiable submanifolds \mathcal{M}_1 and \mathcal{M}_2 of \mathbb{R}^n are said to intersect transversally at a point p if $p \in \mathcal{M}_1 \cap \mathcal{M}_2$ and at this point the tangent spaces satisfy $T_p\mathcal{M}_1 + T_p\mathcal{M}_2 = \mathbb{R}^n$.

- (a) Give an example of a surface and a regular curve in \mathbb{R}^3 that intersect at a single point, but in a non-transversal way.
- (b) Show that if S is a surface and C a curve in \mathbb{R}^3 (both regular), which intersect transversally at $0 \in \mathbb{R}^3$, then one can construct a system of local coordinates (u, v, t) in a neighborhood of 0 such that (u, v) are local parameters of the surface S and t a local parameter of the curve C .

- (c) In the same situation as in (b), prove that 0 is an isolated point of the intersection $S \cap C$ (i.e. there exists an open set $\mathcal{V} \subset \mathbb{R}^3$ such that $\mathcal{V} \cap S \cap C = \{0\}$).

Remark: In the above, saying that a curve or a surface is regular means that it is a submanifold of class \mathcal{C}^k , with $k \geq 1$.

8.5 The window of Viviani is the intersection curve of a sphere with a right circular cylinder that passes through the center of the sphere and whose diameter is equal to the radius of the sphere. If the radius of the sphere is 2, we can therefore assume (up to applying an isometry) that Viviani’s window is defined by the equations:

$$x^2 + y^2 + z^2 = 4 \quad \text{and} \quad (x - 1)^2 + y^2 = 1.$$

We denote this set by V .

- (a) Show by a geometric argument that there exists a point $q \in V$ such that the complement $V \setminus \{q\}$ is a differentiable submanifold of \mathbb{R}^3 . What are the coordinates of q (we accept a heuristic argument)?
- (b) Prove rigorously from the equations of V that $V \setminus \{q\} \subset \mathbb{R}^3$ is a differentiable submanifold.
- (c) Find a regular parametrization of this curve.

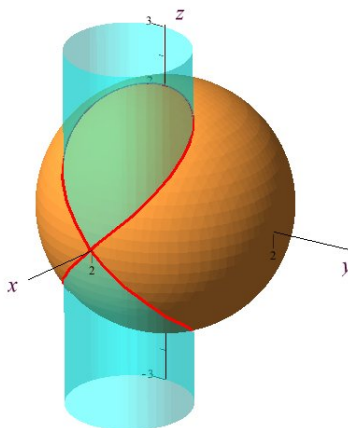


Figure 1: A depiction of Viviani’s window. Credit: Wikipedia.

B. Bonus exercise:

8.6 We have seen in Exercise 7.4 that $O(n)$ and $SL_n(\mathbb{R})$ are submanifolds of $M_n(\mathbb{R})$.

- (a) Describe the tangent space $T_I SL_n(\mathbb{R})$ to the submanifold $SL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ at the point I (the identity matrix).
- (b) Describe the tangent space $T_I O(n)$ to the submanifold $O(n) \subset M_n(\mathbb{R})$ at the point I .